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Nonlinear pure bending of toroidal shells of arbitrary cross-section

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Abstract

The geometrically nonlinear problem of in-plane pure bending of a toroidal shell of arbitrary cross-section (both closed and open) is considered. A finite element algorithm for solution of the problem is proposed. The equilibrium states of the discrete system are determined by an iterative method based on calculation of the coefficients of the first and second variations of the total potential energy. Nonlinear deformation of cylindrical and toroidal shells of closed cross-section is considered. As an example of open cross-sectional contour, a solution for the problem of pure bending of a thin plate is given. The resultant solutions are compared with those of the other authors.

Keywords: Torodial shell; Pure bending; Large elastic displacements; Limiting moment

1. Introduction

The problem of pure bending of toroidal shells arises in the pipe bend analysis. Historically, Dubyaga (1909) gave, by suggestion of Prandtl, the earliest analysis of stresses in curved tubes subjected to pure bending. More careful investigations of stresses and flexibility of thin-walled tubes were carried out by Karman (1911) and Lorenz (1912), who gave theoretical explanation of the experimental results of Bantlin (1910) on flexibility of Ω-shaped pipelines. In these pioneering studies, the problem was formulated and approximate solutions within the framework of small elastic displacements were found. In a series of papers (see e.g. Karl (1943), Beskin (1945), Clark and Reissner (1951), Cheng and Thailer (1970)), the refined solutions to the linear problem were obtained on the basis of different approaches and stresses and flexibility for wide range of geometrical parameters of curved tubes were studied in detail.

Using the assumptions analogous to those adopted by Karman (1911), Brazier (1927) treated the problem of nonlinear deformation of long elastic cylindrical tubes under pure bending and found the value

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of the limiting bending moment for which the instability occurs due to the flattening of the cross-section. Subsequently, Chwalla (1933), Heck (1937), and Konovalov (1940) attempted to refine the value of the limiting moment and gave results differed considerably from those of Brazier (1927). Reissner (1961) reconsidered the problem and reduced it to a fourth-order system of two nonlinear differential equations, which was solved by analytical and numerical methods by Reissner and Weinitschke (1963, 1966), Perrone and Kao (1971), Na and Turski (1974) and Thurston (1965, 1977). The effect of initial ellipticity of cross-section on stability of cylindrical shells was studied by Spence and Toh (1979).

Nonlinear equations for the finite bending of curved tubes of circular cross-section were proposed by Reissner (1959). An alternate variant of the equations and their approximate solutions in the form of power series expansion were given by Axelrad (1960, 1961). An energy based solution was obtained by Kostovetskii (1960). Formulation of the tube bending problem was re-examined by Boyle (1981) from the standpoint of the geometrically nonlinear theory of shells. A sixth-order system of differential equations was derived and solved numerically for the case of circular cross-section. Reissner (1981) showed that by appropriate choice of primary unknowns, the problem can be simplified by reducing to a fourth-order system, from which the well-known linear equations of curved tubes (see e.g. Clark and Reissner (1951)) and equations of finite pure bending of cylindrical shells (Reissner, 1961) follow as special cases. However, as far as the authors are aware, neither analytical nor numerical solutions of this system have been given in the literature so far.

Analysis of the studies dealing with the tube bending problem shows that in most of them tubes of circular cross-section were considered. The existing analytical solutions are restricted to the case of tubes with small initial curvature of the axis and do not make it possible to study the finite bending involving significant flattening of the cross-section. Investigation of nonlinear deformation of tubes, even within the framework of the pure bending model, is a sufficiently complicated problem, which can be solved effectively only by numerical methods. It is therefore of interest to develop a numerical algorithm applicable for wide range of geometrical parameters of tubes and for significant cross-sectional distortion due to bending.

Here we consider a refined numerical algorithm proposed by Kuznetsov and Levyakov (1992) and give some results for tubes of noncircular cross-sections.

2. Strain relations

We consider a sector of thin-walled toroidal shell bent in the plane of curvature of its axial line with the end moments M. Let the shape of cross-section (meridian) be defined in the parametric form $x_i = x_i(s)$, where s is the arc length and $i = 1, 2$. We assume that the cross-sections which are normal to the axial line remain plane and normal to the axial line in the process of loading the shell, but can deform in their planes. The stresses and strains do not change along the shell axis and depend only on the meridional coordinate s. Arbitrary displacements and rotations are allowed and strains are assumed to be small compared to unity. Using the above-mentioned assumptions and the Kirchhoff–Love hypotheses, we write the equation of a shell surface in its initial and deformed state in the vector form

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{e}_i(x_i + z\lambda_i^n), \qquad \mathbf{R}^* = \mathbf{R}_0^* + \mathbf{e}_i^*(x_i^* + z\lambda_i^{n*}), \tag{1}$$

where $\mathbf{R}_0$ is the radius-vector of the axial line, $\mathbf{e}_i = \mathbf{e}_i(t)$ are the unit orthogonal vectors lying in the plane of the cross-section, t is the arc length of the axial line of the shell, λ_i^n are the direction cosines of the normal vector to the middle surface of the shell, z is the normal coordinate to the middle surface of the shell, and the asterisk denotes quantities which refer to the deformed state. Here and henceforth the rule of summation over repeated indices is employed unless otherwise specified.

Using Eq. (1), we obtain the relations for the strains and the curvature changes of the middle surface of the shell in meridional and axial directions:

$$\varepsilon_s = \tfrac{1}{2}(x_i^{*\prime}x_i^{*\prime} - 1), \qquad \kappa_s = x_i^{*\prime}\lambda_i^{n*\prime} - x_i^{\prime}\lambda_i^{n\prime}, \tag{2a}$$

$$\varepsilon_t = A_t^{-1}(\varepsilon + k^* x_1^* - k x_1), \qquad \kappa_t = A_t^{-1}(k^*\lambda_1^{n*} - k\lambda_1^{n}), \tag{2b}$$

where $A_t = 1 + kx_1$ is the Lame parameter, ε and k are the strain and curvature of the axial line, respectively, and the prime denotes derivative with respect to the coordinate s.

3. Energy and equilibrium conditions

The strain energy of the toroidal shell with unit length of the axial line has the form

$$\Pi = \frac{1}{2}\int (T_t\varepsilon_t + T_s\varepsilon_s + M_t\kappa_t + M_s\kappa_s)A_t\,\mathrm{d}s, \tag{3}$$

where integration is taken over the entire cross-section and T_s, T_t, M_s, and M_t are the forces and the bending moments which for the case of an isotropic, linear-elastic body are connected with the strains and the curvatures changes (2) by the relations

$$\begin{aligned} &T_s = B(\varepsilon_s + \nu\varepsilon_t), \quad T_t = B(\varepsilon_t + \nu\varepsilon_t), \quad B = Eh/(1-\nu^2), \quad M_s = D(\kappa_s + \nu\kappa_t),\\ &M_t = D(\kappa_t + \nu\kappa_s), \quad D = Bh^2/12, \end{aligned} \tag{4}$$

in which E is Young's modulus, ν is Poisson's ratio, and h is the thickness of the shell.

The total potential energy has the form $U = \Pi + W$, where $W = -M(k^* - k)$ is the potential of the external bending moments. Using the stationary condition for the total potential energy $\delta U = 0$, we obtain the following system of nonlinear equations of equilibrium:

$$\delta x_1^*\colon H' - k^* T_t = 0, \qquad \delta x_2^*\colon V' = 0, \tag{5}$$

$$\delta\varepsilon\colon \int T_t\,\mathrm{d}s = 0, \qquad \delta k^*\colon \int (T_t + M_t\lambda_1^{n*})\,\mathrm{d}s = M \tag{6}$$

and the boundary conditions on the boundaries of the open cross-sectional contour:

$$\begin{aligned} &\delta x_1^* = 0 \quad \text{or} \quad H = 0,\\ &\delta x_2^* = 0 \quad \text{or} \quad V = 0,\\ &\delta\varphi^* = 0 \quad \text{or} \quad M_s = 0. \end{aligned} \tag{7}$$

Here

$$\begin{aligned} H &= T_s A_t x_1^{*\prime} + (A_s^*)^{-1}(M_s A_t x_2^{*\prime})' - (A_s^*)^{-3}k^* M_t x_1^{*\prime}x_2^{*\prime},\\ V &= T_s A_t x_2^{*\prime} - (A_s^*)^{-1}(M_s A_t x_1^{*\prime})' + (A_s^*)^{-3}k^* M_t (x_1^{*\prime})^2,\\ A_s^* &= (x_i^{*\prime}x_i^{*\prime})^{1/2} \approx 1 + \varepsilon_s,\\ \varphi^* &= -a\tan(x_1^{*\prime}/x_2^{*\prime}). \end{aligned}$$

Eq. (6) express conditions of pure bending, namely, the principal vector is zero and the principal moment is equal to the specified bending moment. These equations can also be considered as boundary conditions at the loaded ends of the shell which are satisfied in Saint-Venant fashion.

In the case of a closed contour, conditions (7) are to be replaced by the periodicity conditions.

It is obvious that exact analytical solutions of the boundary-value problems (5)–(7) can be obtained only in particular cases. To overcome the difficulties arising in direct integration of the above equations, we resort to a numerical method.

4. Numerical algorithm for solution of the problem

We divide the shell into finite elements having length l in the meridional direction. Assuming that the elements are small, we derive an approximate version of relations (2). To this end, we introduce the local coordinate system (α, ζ) with the unit orthogonal vectors $\mathbf{e}_\alpha$ and $\mathbf{e}_\zeta$, attached to the element (Fig. 1). The functions $\zeta(\alpha)$ and $\zeta^*(\alpha)$ which describe the shapes of the element in the initial and deformed states are expanded into Taylor series in the neighborhood of node 1 as follows:

$$\zeta = \tfrac{1}{2}\zeta_1''(\alpha^2 - \alpha l) + \mathrm{O}(l^3), \qquad \zeta^* = \tfrac{1}{2}\zeta_1^{*\prime\prime}(\alpha^2 - \alpha l) + \mathrm{O}(l^3). \tag{8}$$

The element coordinates and direction cosines of the normal vector $\mathbf{n}$ are written in the form (no summation over α)

$$x_i = x_i^p + \lambda_i^\zeta \zeta, \qquad \lambda_i^n = (\lambda_i^\zeta - \lambda_i^\alpha \zeta_{,\alpha}) \Big/ \sqrt{1 + \zeta_{,\alpha}^2}, \tag{9}$$

where x_i^p are the coordinates of the points on the straight line passing through the element nodes and λ_i^α and λ_i^ζ are the direction cosines of the vectors $\mathbf{e}_\alpha$ and $\mathbf{e}_\zeta$, respectively, and the comma denotes differentiation with respect to α. Similarly, for a deformed state, we have

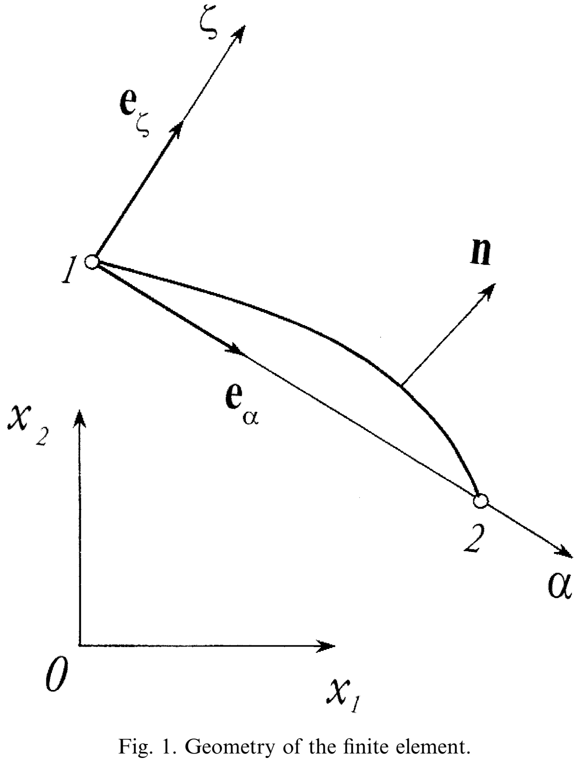

Fig. 1. Geometry of the finite element.

$$x_i^* = x_i^{p*} + \lambda_i^{\zeta*}\zeta^*, \qquad \lambda_i^{n*} = (\lambda_i^{\zeta*} - \lambda_i^{\alpha*}\zeta^*_{,\alpha})\Big/\sqrt{1+\zeta^{*2}_{,\alpha}}. \tag{10}$$

In view of the element smallness, differentiation with respect to the coordinate s can be replaced by differentiation with respect to α. Substituting Eqs. (9) and (10) into Eq. (2a) and using expansion (8) to estimate the leading terms, we arrive at the relations (no summation over p and α)

$$\varepsilon_s = \tfrac{1}{2}(x^{p*}_{i,\alpha}x^{p*}_{i,\alpha} - 1), \qquad \kappa_s = -w_{,\alpha\alpha}, \tag{11}$$

which approximate the initial strain relations (2a) with an accuracy up to terms of order $O(l^2)$. Here $w = \zeta^* - \zeta$ is the deflection.

The functions x_i^p, x_i^{p*}, and w are expressed in terms of nodal parameters as follows:

$$\begin{aligned} &x_i^p = L_j x_{ij}, \quad x_i^{p*} = L_j x^*_{ij}, \quad w = N_i\theta_i, \quad \theta_i = b_k(x^*_{jk}\lambda^{n*}_{ji} - x_{jk}\lambda^n_{ji}),\\ &L_1 = 1 + b_1\alpha, \quad L_1 = b_2\alpha, \qquad b_1 = -l^{-1}, \ b_2 = l^{-1},\\ &N_1 = (\alpha^3 - 2l\alpha^2 + l^2\alpha)l^{-2}, \qquad N_2 = (\alpha^3 - l\alpha^2)l^{-2}, \end{aligned} \tag{12}$$

where x_{ij} and λ^n_{ij} $(i,j = 1,2)$ are the ith coordinates and direction cosines of the normal vector at the jth node of the element, respectively, θ_i is the elastic component of the rotation of the normal vector at the ith node, and L_i and N_i are the shape functions. Substituting Eq. (12) into Eq. (11) and averaging the functions x_1, x_1^*, λ_1^n, and λ_1^{n*} in Eq. (2b) over the element, we obtain the following finite-element relations for the strains and curvature changes:

$$\begin{aligned} &\varepsilon_s = \tfrac{1}{2}(b_i b_k x^*_{ji}x^*_{jk} - 1), \qquad \kappa_s = N_{i,\alpha\alpha}\theta_i,\\ &\varepsilon_t = A_t^{-1}(\varepsilon + k^*x_1^* - kx_1), \qquad \kappa_t = A_t^{-1}(k^*\lambda_1^{n*} - k\lambda_1^n), \end{aligned} \tag{13}$$

in which

$$\begin{aligned} &x_1 = \tfrac{1}{2}(x_{11} + x_{12}), \qquad x_1^* = \tfrac{1}{2}(x^*_{11} + x^*_{12}),\\ &\lambda_1^n = \tfrac{1}{2}(\lambda^n_{11} + \lambda^n_{12}), \qquad \lambda_1^{n*} = \tfrac{1}{2}(\lambda^{n*}_{11} + \lambda^{n*}_{12}). \end{aligned}$$

We introduce the five-component vector of the generalized elastic displacements

$$\mathbf{u}^{\mathrm{T}} = |\,\varepsilon_s, \quad \theta_1, \quad \theta_2, \quad \varepsilon_t, \quad \kappa_t\,|. \tag{14}$$

Substituting relations (4) and (13) into Eq. (3) and integrating between the limits 0 and l, we obtain the strain energy of an element in the form $\Pi = (1/2)\mathbf{u}^{\mathrm{T}}\mathbf{K}\mathbf{u}$, where $\mathbf{K}$ is the 5×5 symmetric stiffness matrix whose nonzero coefficients are given by the expressions

$$\begin{aligned} &K_{11} = BA_t l, \quad K_{14} = \nu K_{11}, \quad K_{22} = Dl^{-1}[4 + k(3x_{11} + x_{12})], \quad K_{23} = 2DA_t l^{-1},\\ &K_{25} = -\nu D(1 + kx_{11}), \quad K_{33} = Dl^{-1}[4 + k(x_{11} + 3x_{12})], \quad K_{35} = \nu D(1 + kx_{12}), \quad K_{44} = K_{11},\\ &K_{55} = DA_t l. \end{aligned}$$

One special feature of the given formulation of the problem is that a finite element of the shell contains the following both nodal and nonnodal unknowns which form the vector of the generalized coordinates $\mathbf{q}$:

$$\mathbf{q}^{\mathrm{T}} = |x^*_{11}, \ x^*_{21}, \ \varphi^*_1, \ x^*_{12}, \ x^*_{22}, \ \varphi^*_2, \ \varepsilon, \ k^*|, \tag{15}$$

where φ^*_i is the angle of rotation of the normal vector at the ith node.

The first and second variations of the strain energy of the element have the form

$$\delta\Pi = \mathbf{g}^{\mathrm{T}}\delta\mathbf{q}, \qquad \delta^2\Pi = \delta\mathbf{q}^{\mathrm{T}}\mathbf{H}\,\delta\mathbf{q},$$

where

$$\mathbf{g} = \mathbf{u}'\mathbf{P}, \quad \mathbf{P} = \mathbf{K}\mathbf{u}, \quad \mathbf{H} = \mathbf{u}'\mathbf{K}(\mathbf{u}')^{\mathrm{T}} + \mathbf{P}_r\mathbf{u}''_r \quad (r = 1, \ldots, 5).$$

Here **g** and **H** are the gradient and the Hess matrix of the total potential energy and $\mathbf{u}'$ and $\mathbf{u}''_r$ are the 8×5 and 8×8 matrices, respectively, which contain the first and second derivatives of the components of the vector **u** with respect to the generalized coordinates (15).

Nonzero components of the matrix $\mathbf{u}'$ have the form (no summation over i)

$$\frac{\partial \varepsilon_s}{\partial x^*_{ij}} = b_j b_k x^*_{jk}, \qquad \frac{\partial \theta_i}{\partial x^*_{jk}} = b_k \lambda^{n*}_{ji}, \qquad \frac{\partial \theta_i}{\partial \varphi^*_i} = b_k x^*_{jk} \lambda^*_{ji}, \qquad \frac{\partial \varepsilon_t}{\partial x^*_{1j}} = \frac{1}{2} k^* A_t^{-1},$$
$$\frac{\partial \varepsilon_t}{\partial \varepsilon} = A_t^{-1}, \qquad \frac{\partial \varepsilon_t}{\partial k^*} = x^*_1 A_t^{-1}, \qquad \frac{\partial \kappa_t}{\partial \varphi^*_i} = \frac{1}{2} k^* \lambda^*_{1i} A_t^{-1}, \qquad \frac{\partial \kappa_t}{\partial k^*} = \lambda^{n*}_{1i} A_t^{-1},$$

where λ^*_{ji} are the direction cosines of the unit vector tangent to the deformed cross-sectional contour at the ith node. The matrix $\mathbf{u}'$ is introduced in such a manner that its rth row corresponds to the component $\mathbf{q}_r$ and its sth column corresponds to the component $\mathbf{u}_s$, i.e., $(\mathbf{u}')_{rs} = \partial u_s / \partial q_r$.

The subscript at the matrices $\mathbf{u}''_r$ indicates which component of the vector **u** is differentiated twice. Nonzero components appearing in matrices $\mathbf{u}''_r$ are calculated by the formulas (no summation over i)

$$\frac{\partial^2 \varepsilon_s}{\partial x^*_{ij}\, \partial x^*_{ik}} = b_j b_k, \qquad \frac{\partial^2 \theta_i}{\partial \varphi^{*2}_i} = -b_k x^*_{jk} \lambda^{n*}_{ji}, \qquad \frac{\partial^2 \theta_i}{\partial x^*_{jk}\, \partial \varphi^*_i} = b_k \lambda^*_{ji},$$
$$\frac{\partial^2 \varepsilon_t}{\partial x^*_{1j}\, \partial k^*} = \frac{1}{2} A_t^{-1}, \qquad \frac{\partial^2 \kappa_t}{\partial \varphi^*_i\, \partial k^*} = \frac{1}{2} \lambda^*_{1i} A_t^{-1}, \qquad \frac{\partial^2 \kappa_t}{\partial \varphi^{*2}_i} = -\frac{1}{2} \lambda^{n*}_{1i} k^* A_t^{-1}.$$

To find a deformed state of the shell, we use the arc-length control method (see e.g. Yang and Shieh (1990)) based on the stepwise determination of a solution. For a certain step, the Newton–Raphson system of equations has the form

$$\mathbf{H}^{k-1} \delta \mathbf{q}^k + \mathbf{w}^{k-1} \delta M^k + \mathbf{g}^{k-1} = 0, \tag{16}$$

where **H** and **g** are the gradient and Hess matrix of the finite element assemblage, **w** is a vector composed of derivatives $\partial^2 W / \partial q_i\, \partial M$, M is the bending moment which is assumed to be an independent variable, and the superscript denotes the iteration number. We note that the superscript at **w** can be omitted, since this vector has the only one nonzero component equal to 1.

System (16) is supplemented by the control equations

$$\begin{aligned} &(\delta \mathbf{q}^1)^{\mathrm{T}} \delta \mathbf{q}^1 + (\delta M^1)^2 = (\delta s)^2 \quad &&\text{for } k = 1, \\ &(\delta \mathbf{q}^k)^{\mathrm{T}} \delta \mathbf{q}^1 + \delta M^1\, \delta M^k = 0 \quad &&\text{for } k > 1, \end{aligned} \tag{17}$$

where δs is a specified quantity representing the arc-length measure of an equilibrium path.

Solution of Eq. (16) can be presented as

$$\delta \mathbf{q}^k = \delta M^k \mathbf{u}^k + \mathbf{v}^k, \tag{18}$$

where the vectors $\mathbf{u}^k$ and $\mathbf{v}^k$ satisfy the systems of equations

$$\mathbf{H}^{k-1} \mathbf{u}^k + \mathbf{w} = 0, \qquad \mathbf{H}^{k-1} \mathbf{v}^k + \mathbf{g}^{k-1} = 0.$$

Substituting Eq. (18) into Eq. (17) and bearing in mind that $\mathbf{g}^0 = 0$ and $\mathbf{v}^1 = 0$, we arrive at the formulas for the load increments

$$\delta M^1 = \pm \delta s \bigg/ \sqrt{1 + (\mathbf{u}^1)^{\mathrm{T}} \mathbf{u}^1} \quad \text{for } k = 1, \tag{19a}$$

$$\delta M^k = -(\mathbf{v}^k)^{\mathrm{T}} \mathbf{u}^1 / [1 + (\mathbf{u}^k)^{\mathrm{T}} \mathbf{u}^1] \quad \text{for } k > 1. \tag{19b}$$

The sign in Eq. (19a) defines the direction of tracing an equilibrium solution path and at ith step it is chosen so that the following condition holds:

$$(\delta \mathbf{q}_i^1)^{\mathrm{T}} \delta \mathbf{q}_{i-1}^1 + \delta M_i^1 \, \delta M_{i-1}^1 > 0.$$

After the quantities $\delta \mathbf{q}^k$ and δM^k are determined, the new values of the unknowns are calculated by means of the following formulas (no summation over j):

$$(x_{ij}^*)^k = (x_{ij}^*)^{k-1} + (\delta x_{ij}^*)^k, \qquad (\lambda_{ij}^{n*})^k = (\lambda_{ij}^{n*})^{k-1} \cos(\delta\varphi_j^*)^k + (\lambda_{ij}^*)^{k-1} \sin(\delta\varphi_j^*)^k,$$
$$\varepsilon^k = \varepsilon^{k-1} + (\delta\varepsilon)^k, \qquad (k^*)^k = (k^*)^{k-1} + (\delta k^*)^k, \qquad M^k = M^{k-1} + (\delta M)^k.$$

The iteration process is continued until the required accuracy of determining unknowns is attained.

To calculate the toroidal shells, one needs as initial data the values of the coordinates and the direction cosines of the normal vector at the nodes of the cross-section under consideration.

It should be noted that setting the curvature k equal to zero, we arrive at the case of cylindrical shell, which can also be studied by the proposed algorithm.

5. Numerical results

5.1. Bending of a thin strip

The first example considered is the problem of flexure of a broad very thin strip. This problem, which admits an exact analytical solution obtained by Lamb (1891), is chosen to demonstrate application of the above algorithm to the case of an open cross-section and also to study the accuracy and convergence of numerical solutions obtained. Calculations were carried out for the case of $\nu = 0.3$ and $2a/h = 100$, $2a$ being the width of the strip. Deflection curves of the strip cross-section are shown in Fig. 2 for various

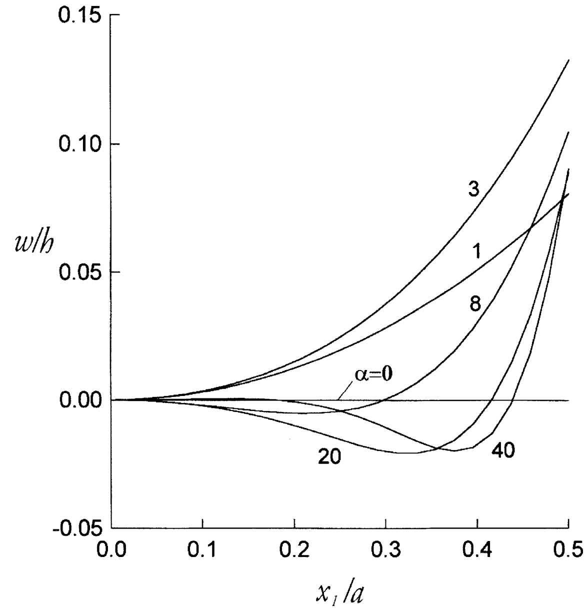

Fig. 2. Deflection curves for a thin strip in pure bending.

Table 1
Convergence study for the problem of strip flexure

α	w/h			
	$N=2$	$N=4$	$N=8$	$N=12$
1	0.08381	0.08168	0.08113	0.08103
2	0.13620	0.12600	0.12362	0.12318
3	0.15568	0.13796	0.13420	0.13352
4	0.15570	0.13504	0.13101	0.13030
8	0.11478	0.10650	0.10545	0.10527
12	0.08392	0.09220	0.09379	0.09404
16	0.06517	0.08663	0.08965	0.09009
20	0.05303	0.08497	0.08854	0.08901
40	0.02727	0.09124	0.09083	0.09075

values of the curvature parameter $\alpha = \sqrt{3(1-\nu^2)}k^*a^2/h$. The results illustrating the convergence of numerical solution are given in Table 1, where the values of deflection w of the strip edge as a function of the curvature parameter are recorded. It is seen that numerical solution converges rapidly as the number of finite elements N increases. For large values of α, the deflection approaches the value $w = 0.09075h$, which agrees with the analytical solution $w = \nu h/\sqrt{12(1-\nu^2)} = 0.09078h$.

5.2. Bending of tubes of circular cross-section

We consider the problem of bending of tubes (toroidal shells) of circular cross-section, characterized by various values of the curvature parameter $\mu = \sqrt{12(1-\nu^2)}kr^2/h$ with $\nu = 0.3$ and $r/h = 100$, where r is the radius of cross-section. Fig. 3 shows the dependencies between the dimensionless parameters of bending

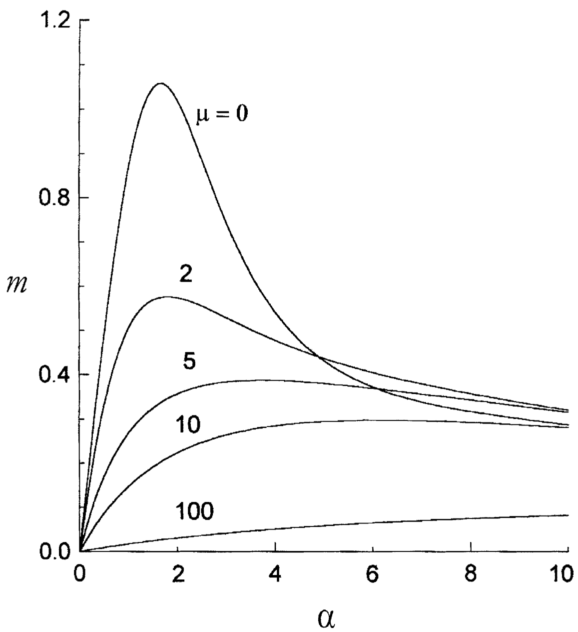

Fig. 3. Dimensionless moment–curvature change relations for tubes of circular cross-section.

Table 2
Critical moment and curvature change parameters for cylindrical shell

Solutions compared	m_c	α_c
Brazier (1927)	1.088	1.633
Chwalla (1933)	1.252	2.664
Konovalov (1940)	1.345	2.532
Reissner and Weinitschke (1966)	1.06	1.66
Perrone and Kao (1971)	1.0568	1.65
Na and Turski (1974)	1.0404	1.650
Thurston (1977)	1.0570968	1.6493953
Present analysis	1.0604	1.66

Table 3
Critical bending moment for curved tubes

λ	m_c	
	Boyle (1981)	Present analysis
0.5	0.36	0.351
1	0.47	0.459
2	0.63	0.628
5	0.84	0.841
10	0.93	0.941

moment $m = \sqrt{12(1-\nu^2)}Mr^2/hEI$ ($I = \pi r^3 h$ is the moment of inertia of the cross-section) and curvature change of the axial line $\alpha = \sqrt{12(1-\nu^2)}(k^* - k)r^2/h$. It is seen that critical bending moment rapidly decreases with increasing μ. Characteristically, the limiting point for large values of $\mu > 10$ is not very pronounced.

The critical parameters m_c and α_c calculated for the cylindrical shell ($\mu = 0$) using the above algorithm are given in Table 2 and compared with those obtained by the other authors. Our results are very close to the values of m_c and α_c determined by solving the nonlinear equations of Reissner (1961) with the use of different numerical techniques. It is interesting to note that the simplest solution given by Brazier (1927) provides good accuracy in describing nonlinear behaviour of the tube up to the loss of stability. Solutions of Chwalla (1933) and Konovalov (1940) significantly overestimate the critical moment and curvature change.

Table 3 lists the values of the critical bending moment for the curved tubes characterized by the parameter $\lambda = \sqrt{12(1-\nu^2)}/\mu$. It should be noted that the Boyle's results (1981) that are given in the table are approximate since they were taken from the graphs. Nevertheless, good correspondence between the numerical solutions compared is observed.

In Figs. 4 and 5, we show on a real scale the forms of flattened cross-sections for $\mu = 5$ and $\mu = 100$, respectively. For large values of μ, cross-sections flatten mainly due to deformation in the neighbourhood of zero Gaussian curvature of the shell. The remaining part of the section is almost undeformed and stress free.

5.3. Bending of tubes of square cross-section

A linear solution to the problem of pure bending of curved thin-walled tubes of rectangular cross-section was given by Timoshenko (1923). However, as far as the authors are aware, this problem has not been treated in geometrically nonlinear formulation.

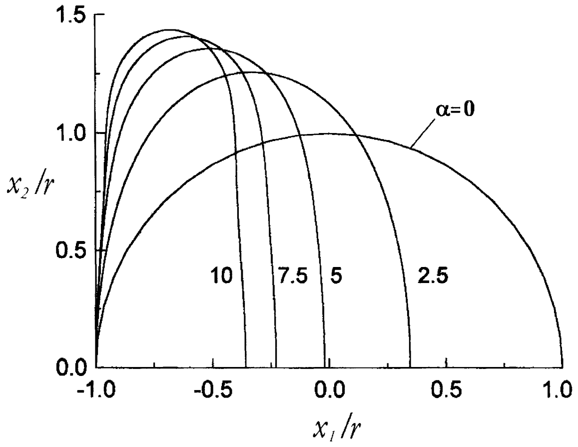

Fig. 4. Flattened circular cross-sections for $\mu = 5$.

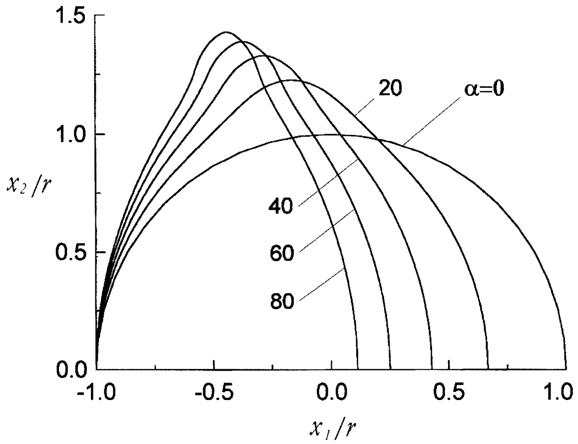

Fig. 5. Flattened circular cross-sections for $\mu = 100$.

We confine our analysis to a square cross-section with side a. The centroidal moment of inertia of the cross-section is $I = (2/3)a^3h$. Fig. 6 shows the parameter of bending moment $m = \sqrt{12(1-\nu^2)}Ma^2/hEI$ versus curvature change parameter $\alpha = \sqrt{12(1-\nu^2)}(k^* - k)a^2/h$ for various values of the tube parameter $\mu = \sqrt{12(1-\nu^2)}ka^2/h$. Results were obtained for $a/h = 100$, $\nu = 0.3$, and division of the half of the cross-section into 32 equal finite elements. Deformed cross-sections for $\mu = 100$ are shown in Fig. 7. One can see that in bending of the tubes characterized by large values of the parameter μ, the cross-section flattens due to bending of the sides that are parallel to the plane of bending; the other two sides remain almost undeformed except for small regions in the neighborhood of the corner points.

6. Conclusions

A geometrically nonlinear formulation of the problem of pure bending of toroidal shells has been re-examined. A numerical algorithm has been developed for determining deformations and stresses of toroidal

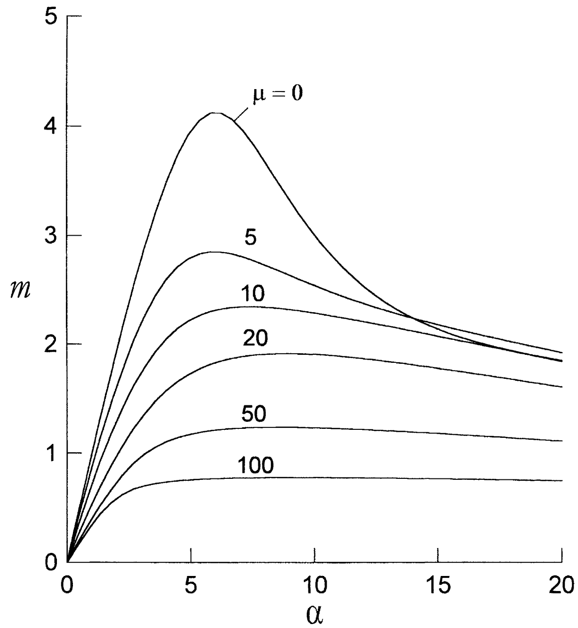

Fig. 6. Dimensionless moment–curvature change relations for tubes of square cross-section.

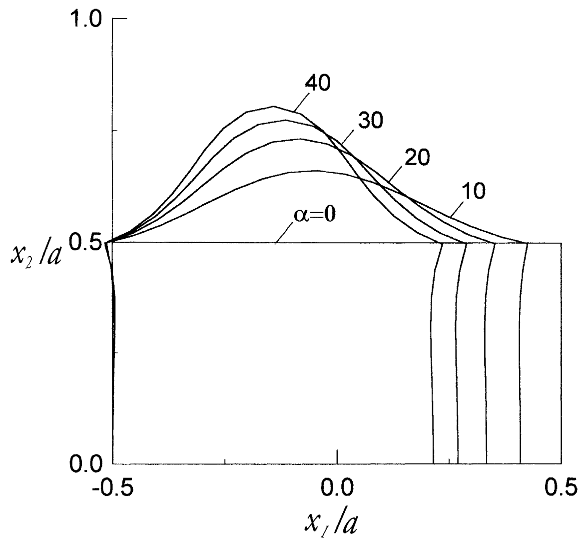

Fig. 7. Flattened square cross-section for $\mu = 100$.

shells in the field of large elastic displacements. Numerical results have been presented for shells of circular and noncircular cross-sections and a broad very thin plate and compared with available solutions. In particular, the value of critical bending moment for cylindrical tubes has been refined and found to be very close to that based on the Reissner's equations (1961). The algorithm is effective, provides high accuracy of calculations, and is applicable for linear and nonlinear analysis of elastic shells.

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